Mathematics: Analysis and Approaches HL

One-page formula sheet Need help? turtlemaths.com/book

Scan to book

Topic 1: Number and algebra

1.2 nth term of an arithmetic sequence: $u_n = u_1 + (n-1)d$

Sum of n terms of an arithmetic sequence: $S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$

1.3 nth term of a geometric sequence: $u_n = u_1 r^{n-1}$

Sum of n terms of a finite geometric sequence: $S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$

1.8 Sum of an infinite geometric sequence: $S_\infty = \frac{u_1}{1 - r}, \quad |r| < 1$

1.4 Compound interest: $FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$

1.5 Exponents and logarithms ($a, b > 0$, $a \neq 1$): $a^x = b \iff x = \log_a b$

1.7 Laws of logarithms: $\log_a xy = \log_a x + \log_a y; \quad \log_a \frac{x}{y} = \log_a x - \log_a y$

$\log_a x^m = m \log_a x; \quad \log_a x = \frac{\log_b x}{\log_b a}$

Exponential and logarithmic functions: $a^x = \mathrm{e}^{x \ln a}; \quad \log_a a^x = x = a^{\log_a x}$

1.9 Binomial theorem, $n \in \mathbb{N}$: $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$

1.10 Combinations and permutations: $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}; \quad {}^nP_r = \frac{n!}{(n-r)!}$

Extension of the binomial theorem, $n \in \mathbb{Q}$: $(a+b)^n = a^n \left(1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \cdots\right)$

1.12 Complex numbers: $z = a + b\mathrm{i}$

1.13 Modulus-argument (polar) and Euler form: $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta} = r\,\mathrm{cis}\,\theta$

1.14 De Moivre's theorem: $[r(\cos\theta + \mathrm{i}\sin\theta)]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = r^n\mathrm{e}^{\mathrm{i}n\theta}$

Topic 2: Functions

2.1 Equations of a straight line: $y = mx + c; \quad ax + by + d = 0; \quad y - y_1 = m(x - x_1)$

Gradient formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$

2.6 Axis of symmetry of a quadratic: $f(x) = ax^2 + bx + c \Rightarrow x = -\frac{b}{2a}$

2.7 Solutions of a quadratic equation: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \ a \neq 0; \qquad \Delta = b^2 - 4ac$

2.12 Sum and product of the roots of a polynomial: $\text{sum} = -\frac{a_{n-1}}{a_n}; \qquad \text{product} = \frac{(-1)^n a_0}{a_n}$

Topic 3: Geometry and trigonometry – prior learning

Area of a parallelogram: $A = bh \quad (b \text{ base}, \ h \text{ height})$

Area of a triangle: $A = \frac{1}{2}bh$

Area of a trapezoid: $A = \frac{1}{2}(a+b)h \quad (a, b \text{ parallel sides})$

Area of a circle: $A = \pi r^2$

Circumference of a circle: $C = 2\pi r$

Volume of a cuboid: $V = lwh$

Volume of a cylinder: $V = \pi r^2 h$

Volume of a prism: $V = Ah \quad (A \text{ cross-section})$

Curved surface area of a cylinder: $A = 2\pi rh$

Distance between (x_1, y_1) and (x_2, y_2): $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Midpoint of (x_1, y_1) and (x_2, y_2): $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Topic 3: Geometry and trigonometry

3.1 Distance between (x_1, y_1, z_1) and (x_2, y_2, z_2): $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Midpoint of (x_1, y_1, z_1) and (x_2, y_2, z_2): $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

Volume of a right pyramid: $V = \frac{1}{3}Ah \quad (A \text{ base area})$

Volume of a right cone: $V = \frac{1}{3}\pi r^2 h$

Curved surface area of a cone: $A = \pi rl \quad (l \text{ slant height})$

Volume of a sphere: $V = \frac{4}{3}\pi r^3$

Surface area of a sphere: $A = 4\pi r^2$

3.2 Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Area of a triangle: $A = \frac{1}{2}ab\sin C$

3.4 Length of an arc (θ in radians): $l = r\theta$

Area of a sector (θ in radians): $A = \frac{1}{2}r^2\theta$

3.5 Identity for $\tan\theta$: $\tan\theta = \frac{\sin\theta}{\cos\theta}$

3.6 Pythagorean identity: $\cos^2\theta + \sin^2\theta = 1$

Double angle identities: $\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

3.9 Reciprocal trigonometric identities: $\sec\theta = \frac{1}{\cos\theta}; \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$

Pythagorean identities: $1 + \tan^2\theta = \sec^2\theta; \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

3.10 Compound angle identities: $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

Double angle identity for tan: $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

3.12 Magnitude of a vector: $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

3.13 Scalar product: $\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3 = |\mathbf{v}||\mathbf{w}|\cos\theta$

Angle between two vectors: $\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|\mathbf{v}||\mathbf{w}|}$

3.14 Vector equation of a line: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

Parametric and Cartesian forms: $x = x_0 + \lambda l, \ y = y_0 + \lambda m, \ z = z_0 + \lambda n$

$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$

3.16 Vector product: $\mathbf{v}\times\mathbf{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$

$|\mathbf{v}\times\mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin\theta$

Area of a parallelogram: $A = |\mathbf{v}\times\mathbf{w}|$

3.17 Vector equation of a plane: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$

Normal and Cartesian forms: $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}; \qquad ax + by + cz = d$

Topic 4: Statistics and probability

4.2 Interquartile range: $\mathrm{IQR} = Q_3 - Q_1$

4.3 Mean of a data set: $\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}, \qquad n = \sum_{i=1}^{k} f_i$

4.5 Probability of an event A: $\mathrm{P}(A) = \frac{n(A)}{n(U)}$

Complementary events: $\mathrm{P}(A) + \mathrm{P}(A') = 1$

4.6 Combined events: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

Mutually exclusive events: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

Conditional probability: $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Independent events: $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

4.7 Expected value of a discrete random variable X: $\mathrm{E}(X) = \sum_{i=1}^{k} x_i\,\mathrm{P}(X = x_i)$

4.8 Binomial distribution: $X \sim \mathrm{B}(n, p)$

Mean and variance: $\mathrm{E}(X) = np, \qquad \mathrm{Var}(X) = np(1-p)$

4.12 Standardized normal variable: $z = \frac{x - \mu}{\sigma}$

4.13 Bayes' theorem: $\mathrm{P}(B \mid A) = \frac{\mathrm{P}(B)\mathrm{P}(A \mid B)}{\mathrm{P}(B)\mathrm{P}(A \mid B) + \mathrm{P}(B')\mathrm{P}(A \mid B')}$

$\mathrm{P}(B_i \mid A) = \frac{\mathrm{P}(B_i)\mathrm{P}(A \mid B_i)}{\sum_j \mathrm{P}(B_j)\mathrm{P}(A \mid B_j)}$

4.14 Variance and standard deviation: $\sigma^2 = \frac{\sum_{i=1}^{k} f_i(x_i - \mu)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$

$\sigma = \sqrt{\frac{\sum_{i=1}^{k} f_i(x_i - \mu)^2}{n}}$

Linear transformation of a random variable: $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b; \quad \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Expected value, continuous X: $\mathrm{E}(X) = \mu = \int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x$

Variance: $\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2] = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

Variance, discrete and continuous X: $\mathrm{Var}(X) = \sum(x - \mu)^2\mathrm{P}(X = x) = \sum x^2\mathrm{P}(X = x) - \mu^2$

$\mathrm{Var}(X) = \int_{-\infty}^{\infty}(x - \mu)^2 f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x - \mu^2$

Topic 5: Calculus

5.12 Derivative from first principles: $f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$

5.3 Derivative of x^n: $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

5.6 Standard derivatives: $f(x) = \sin x \Rightarrow f'(x) = \cos x$

$f(x) = \cos x \Rightarrow f'(x) = -\sin x$

e^x and $\ln x$: $f(x) = \mathrm{e}^x \Rightarrow f'(x) = \mathrm{e}^x$

$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

Chain rule: $y = g(u), \ u = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$

Product rule: $y = uv \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

Quotient rule: $y = \frac{u}{v} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

5.15 Further standard derivatives: $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$

$f(x) = \sec x \Rightarrow f'(x) = \sec x\tan x$

$f(x) = \operatorname{cosec} x \Rightarrow f'(x) = -\operatorname{cosec} x\cot x$

$f(x) = \cot x \Rightarrow f'(x) = -\operatorname{cosec}^2 x$

$f(x) = a^x \Rightarrow f'(x) = a^x(\ln a)$

$f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x\ln a}$

$f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1 - x^2}}$

$f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1 - x^2}}$

$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1 + x^2}$

5.9 Kinematics: $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$

Distance travelled from t_1 to t_2: $\int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t$

Displacement from t_1 to t_2: $\int_{t_1}^{t_2} v(t)\,\mathrm{d}t$

5.5 Integral of x^n: $\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

5.10 Standard integrals: $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$

$\int \sin x\,\mathrm{d}x = -\cos x + C$

$\int \cos x\,\mathrm{d}x = \sin x + C$

$\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + C$

5.15 Further standard integrals: $\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C$

$\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

$\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + C, \ |x| < a$

5.16 Integration by parts: $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

5.11 Area enclosed by a curve and the x-axis: $A = \int_a^b |y|\,\mathrm{d}x$

5.17 Area enclosed by a curve and the y-axis: $A = \int_a^b |x|\,\mathrm{d}y$

Volume of revolution: $V = \int_a^b \pi y^2\,\mathrm{d}x \quad \text{or} \quad V = \int_a^b \pi x^2\,\mathrm{d}y$

5.18 Euler's method: $y_{n+1} = y_n + h\,f(x_n, y_n); \quad x_{n+1} = x_n + h$

Integrating factor for $y' + P(x)y = Q(x)$: $\mathrm{e}^{\int P(x)\,\mathrm{d}x}$

5.19 Maclaurin series: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$

Maclaurin series for special functions:

$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \cdots$

$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$