# Mathematics: Applications and Interpretation HL

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## Topic 1: Number and algebra

**1.2** $n$th term of an arithmetic sequence: $u_n = u_1 + (n-1)d$

Sum of $n$ terms of an arithmetic sequence: $S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$

**1.3** $n$th term of a geometric sequence: $u_n = u_1 r^{n-1}$

Sum of $n$ terms of a finite geometric sequence: $S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$

**1.11** Sum of an infinite geometric sequence: $S_\infty = \frac{u_1}{1 - r}, \quad |r| < 1$

**1.4** Compound interest: $FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$

**1.5** Exponents and logarithms ($a, b > 0$, $a \neq 1$): $a^x = b \iff x = \log_a b$

**1.9** Laws of logarithms: $\log_a xy = \log_a x + \log_a y; \quad \log_a \frac{x}{y} = \log_a x - \log_a y$

$\log_a x^m = m \log_a x$

**1.6** Percentage error: $\varepsilon = \left|\frac{v_A - v_E}{v_E}\right| \times 100\%$

**1.12** Complex numbers: $z = a + bi$

Discriminant: $\Delta = b^2 - 4ac$

**1.13** Modulus-argument (polar) and Euler form: $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta} = r\operatorname{cis}\theta$

**1.14** Determinant of a $2 \times 2$ matrix: $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \mathbf{A} = ad - bc$

Inverse of a $2 \times 2$ matrix: $\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad ad \neq bc$

**1.15** Power formula for a matrix: $\mathbf{M}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$

## Topic 2: Functions

Solutions of a quadratic equation: $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a \neq 0$

**2.1** Equations of a straight line: $y = mx + c; \quad ax + by + d = 0; \quad y - y_1 = m(x - x_1)$

Gradient formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$

**2.5** Axis of symmetry of a quadratic: $f(x) = ax^2 + bx + c \Rightarrow x = -\frac{b}{2a}$

**2.9** Logistic function: $f(x) = \frac{L}{1 + Ce^{-kx}}, \quad L, C, k > 0$

## Topic 3: Geometry and trigonometry – prior learning

Area of a parallelogram: $A = bh \quad (b \text{ base}, \ h \text{ height})$

Area of a triangle: $A = \frac{1}{2}bh$

Area of a trapezoid: $A = \frac{1}{2}(a + b)h \quad (a, b \text{ parallel sides})$

Area of a circle: $A = \pi r^2$

Circumference of a circle: $C = 2\pi r$

Volume of a cuboid: $V = lwh$

Volume of a cylinder: $V = \pi r^2 h$

Volume of a prism: $V = Ah \quad (A \text{ cross-section})$

Curved surface area of a cylinder: $A = 2\pi rh$

Distance between $(x_1, y_1)$ and $(x_2, y_2)$: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Midpoint of $(x_1, y_1)$ and $(x_2, y_2)$: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

## Topic 3: Geometry and trigonometry

**3.1** Distance between $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Midpoint of $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

Volume of a right pyramid: $V = \frac{1}{3}Ah \quad (A \text{ base area})$

Volume of a right cone: $V = \frac{1}{3}\pi r^2 h$

Curved surface area of a cone: $A = \pi rl \quad (l \text{ slant height})$

Volume of a sphere: $V = \frac{4}{3}\pi r^3$

Surface area of a sphere: $A = 4\pi r^2$

**3.2** Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Area of a triangle: $A = \frac{1}{2}ab\sin C$

**3.4** Length of an arc ($\theta$ in degrees): $l = \frac{\theta}{360} \times 2\pi r$

Area of a sector ($\theta$ in degrees): $A = \frac{\theta}{360} \times \pi r^2$

**3.7** Length of an arc ($\theta$ in radians): $l = r\theta$

Area of a sector ($\theta$ in radians): $A = \frac{1}{2}r^2\theta$

**3.8** Identities: $\cos^2\theta + \sin^2\theta = 1; \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$

**3.9** Reflection in $y = (\tan\theta)x$: $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

Stretch parallel to the $x$-axis, factor $k$: $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

Stretch parallel to the $y$-axis, factor $k$: $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

Enlargement, factor $k$, centre $(0, 0)$: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Anticlockwise rotation, angle $\theta$ about origin: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Clockwise rotation, angle $\theta$ about origin: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

**3.10** Magnitude of a vector: $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

**3.11** Vector equation of a line: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

Parametric form: $x = x_0 + \lambda l, \quad y = y_0 + \lambda m, \quad z = z_0 + \lambda n$

**3.13** Scalar product: $\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$

$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$

Angle between two vectors: $\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|\mathbf{v}||\mathbf{w}|}$

Vector product: $\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$

$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin\theta$

Area of a parallelogram: $A = |\mathbf{v} \times \mathbf{w}|$

## Topic 4: Statistics and probability

**4.2** Interquartile range: $\mathrm{IQR} = Q_3 - Q_1$

**4.3** Mean of a data set: $\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}, \qquad n = \sum_{i=1}^{k} f_i$

**4.5** Probability of an event $A$: $\mathrm{P}(A) = \frac{n(A)}{n(U)}$

Complementary events: $\mathrm{P}(A) + \mathrm{P}(A') = 1$

**4.6** Combined events: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

Mutually exclusive events: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

Conditional probability: $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Independent events: $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

**4.7** Expected value of a discrete random variable $X$: $\mathrm{E}(X) = \sum_{i=1}^{k} x_i \, \mathrm{P}(X = x_i)$

**4.8** Binomial distribution: $X \sim \mathrm{B}(n, p)$

Mean and variance: $\mathrm{E}(X) = np, \qquad \mathrm{Var}(X) = np(1 - p)$

**4.14** Linear transformation of a random variable: $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b; \quad \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Linear combinations of independent $X_1, \ldots, X_n$: $\mathrm{E}\left(\sum a_i X_i\right) = \sum a_i \mathrm{E}(X_i)$

$\mathrm{Var}\left(\sum a_i X_i\right) = \sum a_i^2 \mathrm{Var}(X_i)$

Unbiased estimate of population variance: $s_{n-1}^2 = \frac{n}{n - 1}s_n^2$

**4.17** Poisson distribution: $X \sim \mathrm{Po}(m); \quad \mathrm{E}(X) = m, \ \mathrm{Var}(X) = m$

**4.19** Transition matrices: $\mathbf{T}^n\mathbf{s}_0 = \mathbf{s}_n \quad (\mathbf{s}_0 \text{ initial state})$

## Topic 5: Calculus

**5.3** Derivative of $x^n$: $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

**5.9** Standard derivatives: $f(x) = \sin x \Rightarrow f'(x) = \cos x$

$f(x) = \cos x \Rightarrow f'(x) = -\sin x$

$f(x) = \tan x \Rightarrow f'(x) = \frac{1}{\cos^2 x}$

$f(x) = \mathrm{e}^x \Rightarrow f'(x) = \mathrm{e}^x$

$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

Chain rule: $y = g(u), \ u = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$

Product rule: $y = uv \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

Quotient rule: $y = \frac{u}{v} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

**5.5** Integral of $x^n$: $\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n + 1} + C, \quad n \neq -1$

**5.8** The trapezoidal rule: $\int_a^b y \, \mathrm{d}x \approx \frac{1}{2}h\left((y_0 + y_n) + 2(y_1 + \cdots + y_{n-1})\right)$

$h = \frac{b - a}{n}$

**5.11** Standard integrals: $\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + C$

$\int \sin x \, \mathrm{d}x = -\cos x + C$

$\int \cos x \, \mathrm{d}x = \sin x + C$

$\int \frac{1}{\cos^2 x} \, \mathrm{d}x = \tan x + C$

$\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + C$

**5.12** Area enclosed by a curve and an axis: $A = \int_a^b |y| \, \mathrm{d}x \quad \text{or} \quad A = \int_a^b |x| \, \mathrm{d}y$

Volume of revolution: $V = \int_a^b \pi y^2 \, \mathrm{d}x \quad \text{or} \quad V = \int_a^b \pi x^2 \, \mathrm{d}y$

**5.13** Acceleration: $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}s}$

Distance travelled from $t_1$ to $t_2$: $\int_{t_1}^{t_2} |v(t)| \, \mathrm{d}t$

Displacement from $t_1$ to $t_2$: $\int_{t_1}^{t_2} v(t) \, \mathrm{d}t$

**5.16** Euler's method: $y_{n+1} = y_n + h\,f(x_n, y_n); \quad x_{n+1} = x_n + h$

Euler's method for coupled systems: $x_{n+1} = x_n + h\,f_1(x_n, y_n, t_n)$

$y_{n+1} = y_n + h\,f_2(x_n, y_n, t_n), \quad t_{n+1} = t_n + h$

**5.17** Exact solution for coupled linear differential equations: $\mathbf{x} = Ae^{\lambda_1 t}\mathbf{p}_1 + Be^{\lambda_2 t}\mathbf{p}_2$